Concerning the frame of minimal prime ideals of pointfree function rings

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Abstract. Let $L$ be a completely regular frame and $RL$ be the ring of continuous real-valued functions on $L$. We study the frame $\mathcal{O}(\text{Min}(RL))$ of minimal prime ideals of $RL$ in relation to $\beta L$. For $I \in \beta L$, denote by $OI$ the ideal $\{\alpha \in RL \mid \text{coz } \alpha \in I\}$ of $RL$. We show that sending $I$ to the set of minimal prime ideals not containing $OI$ produces a $*$-dense one-one frame homomorphism $\beta L \to \mathcal{O}(\text{Min}(RL))$ which is an isomorphism if and only if $L$ is basically disconnected.

1 Introduction

The study of the space of minimal prime ideals of a commutative ring was initiated by Henriksen and Jerison [13]. In that article they relate the space $\text{Min}(C(X))$ to $\beta X$ by constructing a continuous function $\text{Min}(C(X)) \to \beta X$ which maps no proper closed subset of $\text{Min}(C(X))$.

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onto $\beta X$, and is a homeomorphism precisely when $X$ is basically connected. Our intent in this article is to study the frame $\mathcal{O}(\text{Min}(RL))$ in relation to $\beta L$ and investigate if there are results that parallel the spatial ones we have just mentioned.

We define a map $\tau_L: \beta L \rightarrow \mathcal{O}(\text{Min}(RL))$ by sending an element of $\beta L$ to the set of all minimal prime ideals of $RL$ which do not contain the ideal $\{\alpha \in RL \mid \text{coz} \alpha \in I\}$ of $RL$. This turns out to be a one-one frame homomorphism (Proposition 3.1), which is an isomorphism if and only if $L$ is basically disconnected (Proposition 3.2). This accords with the spatial result of Henriksen and Jerison because a topological space is basically disconnected precisely if its frame of open sets is basically disconnected.

A frame homomorphism is called $\ast$-dense if whenever its right adjoint sends an element to the bottom, then that element is the bottom of the codomain of the homomorphism. This notion generalises the property of a continuous map sending no proper closed subset of its domain onto its codomain. We show in Proposition 3.3 that $\tau_L$ is $\ast$-dense, so that, once again, we have a result which is in agreement with its spatial counterpart.

Every frame homomorphism $h: L \rightarrow M$ between completely regular frames has a Stone extension $h^\beta: \beta L \rightarrow \beta M$, which is a unique frame homomorphism making the square below commute.

$$
\begin{array}{ccc}
\beta L & \xrightarrow{h^\beta} & \beta M \\
\downarrow{j_L} & & \downarrow{j_M} \\
L & \xrightarrow{h} & M \\
\end{array}
$$

For those $h$ for which the ring homomorphism $Rh: RL \rightarrow RM$ contracts minimal prime ideals to minimal prime ideals (for instance whenever $L$ is a $P$-frame), we construct a frame homomorphism $\bar{h}: \mathcal{O}(\text{Min}(RL)) \rightarrow$
\[ \mathcal{O}(\text{Min}(\mathcal{R}M)) \] which makes the square

\[
\begin{array}{ccc}
\beta L & \xrightarrow{h^\beta} & \beta M \\
\downarrow{\tau_L} & & \downarrow{\tau_M} \\
\mathcal{O}(\text{Min}(\mathcal{R}L)) & \xrightarrow{\bar{h}} & \mathcal{O}(\text{Min}(\mathcal{R}M))
\end{array}
\]

commute. If \( L \) is basically disconnected, then \( \bar{h} \) is unique with this property.

2 Preliminaries

All our frames are completely regular, and our main reference for frames is [14]. For a detailed discussion on the ring of continuous real-valued functions on a frame, the reader should also consult [1] and [2]. We denote the right adjoint of a homomorphism \( h: L \to M \) by \( h^* \). A homomorphism is called \textit{dense} if it maps only the bottom element to the bottom element; and it is \textit{codense} if the top is the only element it sends to the top. An element \( p \) of a frame is called a \textit{point} if \( p \neq 1 \) and \( a \land b \leq p \) implies \( a \leq p \) or \( b \leq p \). We denote by \( \text{Pt}(L) \) the set of all points of \( L \). The points of a regular frame are precisely those elements which are maximal strictly below the top. A \textit{complemented} element in a frame is an element which joins its pseudocomplement at the top.

As in [2], we denote by \( \mathcal{R}L \) the ring of all real-valued continuous functions on \( L \). The reader will recall that the underlying set of this ring is the set of all frame homomorphisms \( \mathcal{L}(\mathbb{R}) \to L \), where \( \mathcal{L}(\mathbb{R}) \) denotes the frame of reals. A \textit{cozero element} of \( L \) is an element of the form \( \varphi((-\infty,0) \lor (0,-)) \), for some \( \varphi \in \mathcal{R}L \). An element \( a \) of \( L \) is a cozero element if and only if there is a sequence \( (a_n) \) in \( L \) such that \( a_n \ll a \) for each \( n \) and \( a = \bigvee a_n \). The set of all cozero elements of \( L \) is called the \textit{cozero part} of \( L \) and is denoted by \( \text{Coz} L \). It is a sub-\( \sigma \)-frame of \( L \) which generates \( L \) if \( L \) is completely regular. General properties of cozero elements and cozero parts of frames can be found in [3]. A
homomorphism \( h: L \to M \) is **coz-onto** if for every \( d \in \text{Coz} M \) there is a \( c \in \text{Coz} L \) with \( h(c) = d \). As usual, we denote by \( \beta L \) the Stone-Čech compactification of \( L \), which we take to be the frame of regular ideals of \( \text{Coz} L \). For our purposes this is more convenient than viewing \( \beta L \) as the frame of completely regular ideals of \( L \). The right adjoint of the join map \( j_L: \beta L \to L \) will be denoted by \( r_L \). Because of the way have chosen to view \( \beta L \), the right adjoint \( r_L \) is given by \( r_L(a) = \{ c \in \text{Coz} L \mid c \ll a \} \).

For each \( I \in \beta L \), the ideals \( O^I \) and \( M^I \) of \( RL \) are defined as follows:

\[
O^I = \{ \alpha \in RL \mid r_L(\text{coz} \alpha) \prec I \} \quad \text{and} \quad M^I = \{ \alpha \in RL \mid r_L(\text{coz} \alpha) \leq I \}.
\]

Since for any \( I, J \in \beta L \), \( I \prec J \) implies \( \bigvee I \in J \), it follows that

\[
O^I = \{ \alpha \in RL \mid \text{coz} \alpha \in I \}.
\]

For any \( a \in L \) we abbreviate \( M^{r_L(a)} \) as \( M_a \), and remark that

\[
M_a = \{ \alpha \in RL \mid \text{coz} \alpha \leq a \}.
\]

It is shown in [6, Lemma 3.1] that, for any \( \alpha \in RL \),

\[
\text{Ann} (\alpha) = M_{(\text{coz} \alpha)^*} \quad \text{and} \quad \text{Ann}^2 (\alpha) = M_{(\text{coz} \alpha)^{**}}.
\]

Furthermore, the annihilator ideals of \( RL \) are exactly the ideals \( M_a^* \), for \( a \in L \). The maximal ideals of \( RL \) are precisely the ideals \( M^I \), for \( I \in \text{Pt}(\beta L) \); and for any prime ideal \( P \) of \( RL \), there is a unique \( I \in \text{Pt}(\beta L) \) such that \( O^I \subseteq P \subseteq M^I \). See [5] for the proofs of these assertions.

### 3 The main results

Let us recall what the frame \( \mathcal{D} (\text{Min}(RL)) \) looks like. For any ideal \( Q \) in \( RL \), let \( \mathcal{U}(Q) \) be the set

\[
\mathcal{U}(Q) = \{ P \in \text{Min}(RL) \mid P \nsubseteq Q \}.
\]
For the principal ideal $\langle \alpha \rangle$, we abbreviate $U(\langle \alpha \rangle)$ as $U(\alpha)$. Then

$$\Omega(\text{Min}(RL)) = \{U(Q) \mid Q \text{ is an ideal of } RL\},$$

and the set $\{U(\alpha) \mid \alpha \in RL\}$ is a base for this frame consisting of complemented elements, thus making the frame zero-dimensional, and hence completely regular. We shall denote the bottom of this frame by $\bot$, which of course is the empty set, and its top by $\top$. An ideal $Q$ of $RL$ is called a $z$-ideal if for any $\alpha, \gamma \in RL$, $\text{coz} \alpha = \text{coz} \gamma$ and $\alpha \in Q$ imply $\gamma \in Q$. The equal sign can be replaced with $\leq$. Minimal prime ideals are $z$-ideals. Define the map

$$\tau_L : \beta L \to \Omega(\text{Min}(RL)) \quad \text{by} \quad \tau_L(I) = U(O^I).$$

**Proposition 3.1.** For any completely regular frame $L$, the map $\tau_L$ is a one-one frame homomorphism.

*Proof.* Clearly, $\tau_L$ preserves the bottom and the top. It also preserves binary meets because, for any $I, J \in \beta L$, $O^{I \land J} = O^I \cap O^J$. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a collection of elements of $\beta L$, and, for brevity, write $I = \bigvee_I I_\lambda$. We show that $\tau_L(I) \subseteq \bigcup_{\lambda} \tau_L(I_\lambda)$, which will prove that $\tau_L$ preserves meets since it preserves order. Let $P$ be in $\tau_L(I)$. Then $O^I \nsubseteq P$, and so there is an $\alpha \in O^I$ such that $\alpha \notin P$. By the way joins are calculated in $\beta L$, there are indices $\lambda_1, \ldots, \lambda_n$ in $\Lambda$, and elements $c_i \in I_{\lambda_i}$, for $i = 1, \ldots, n$, such that

$$\text{coz} \alpha = c_1 \lor \cdots \lor c_m.$$  

For each $i$, take a positive $\gamma_i \in RL$ such that $c_i = \text{coz}(\gamma_i)$. Suppose, by way of contradiction, that $P \notin \bigcup_{\lambda} \tau_L(I_\lambda)$. Then $O^{I_\lambda} \subseteq P$ for every $\lambda \in \Lambda$. In particular, $O^{I_{\lambda_i}} \subseteq P$ for each $i = 1, \ldots, n$, which implies $\gamma_i \in P$ for
each $i$, and hence $\gamma_1 + \cdots + \gamma_n \in P$. Since $\text{coz} \alpha = \text{coz}(\gamma_1 + \cdots + \gamma_n)$ and $P$ is a $z$-ideal, we have that $\alpha \in P$, and thus we have reached a contradiction. Therefore $\tau_L$ is a frame homomorphism.

Since the frames $\beta L$ and $\mathcal{O}(\text{Min}(RL))$ are regular, to prove that $\tau_L$ is one-one it suffices to show that $\tau_L$ is codense. Consider therefore any $I \in \beta L$ with $\tau_L(I) = \top$. This implies $U(O^I) = \top$, so that $O^I \not\subseteq P$, for any minimal prime ideal $P$ of $RL$. Suppose, for contradiction, that $I \neq 1_{\beta L}$. Since $\beta L$ has enough points, take a point $J \in \text{Pt}(\beta L)$ with $I \leq J$. The maximal ideal $M^J$ contains a minimal prime ideal, say $P$. Then $O^J \subseteq P$. Since $I \leq J$, we have $O^I \subseteq O^J \subseteq P$; and hence a contradiction. Therefore $I = 1_{\beta L}$, as required. □

Recall that a frame $L$ is basically disconnected if $c^* \lor c^{**} = 1$ for every $c \in \text{Coz} L$. Observe that if $a \in L$ is complemented, then $O^{r_L(a)} = M_a$. This is so because if $\alpha \in M_a$ then $\text{coz} \alpha \leq a \prec\prec a$, so that $\text{coz} \alpha \in r_L(a)$, hence $\alpha \in O^{r_L(a)}$. We will need some results from elsewhere.

For a commutative ring $A$ with identity, let $\text{Max}(A)$ denote the space of maximal ideals of $A$ with the hull-kernel topology. Recall that the topology of $\text{Max}(A)$ is precisely the frame

$$\mathcal{O}(\text{Max}(A)) = \{ M(Q) \mid Q \text{ is an ideal of } A \},$$

where, for any ideal $Q$ of $A$,

$$M(Q) = \{ M \in \text{Max}(A) \mid M \not\subseteq Q \}$$

As before we write $M(a)$ for $M(\langle a \rangle)$. Scott Woodward proved in his PhD thesis [15] that if $A$ is an $f$-ring with zero Jacobson radical, then a subset of $\text{Max}(A)$ is clopen precisely if it is of the form $M(e)$, for some idempotent $e \in A$.

It can be deduced from results in [4] that, for any completely regular
frame $L$,
\[
\mathcal{D}(\text{Max}(RL)) \cong \beta L,
\]
in perfect analogy with the spatial result that $\text{Max}(C(X))$ is homeomorphic to $\beta X$, for any Tychonoff space $X$. For each ideal $Q$ of $RL$, denote by $I_Q$ the element of $\beta L$ given by
\[
I_Q = \bigvee \{r_L(\text{coz } \alpha) \mid \alpha \in Q\}.
\]
A careful analysis reveals that the map
\[
\varrho_L: \mathcal{D}(\text{Max}(RL)) \to \beta L \quad \text{defined by} \quad \varrho_L(M(Q)) = I_Q
\]
is well defined, and is a frame isomorphism. We shall demonstrate only that it is well defined. For this we need only show that if $P$ and $Q$ are ideals of $RL$ with $M(P) = M(Q)$, then $I_P = I_Q$. Observe that, for any $J \in \text{Pt}(\beta L)$,
\[
Q \subseteq M^J \iff r_L(\text{coz } \alpha) \leq J \text{ for every } \alpha \in Q \iff I_Q \leq J.
\]
Since $I_Q$ is the meet of points of $\beta L$ above it, and since $M(P) = M(Q)$ if and only if $P$ and $Q$ are contained in exactly the same maximal ideals, it follows that $I_P = I_Q$.

We remind the reader that an ideal $I$ of a commutative ring is called a $d$-ideal if, for every $a \in I$, $\text{Ann}^2(a) \subseteq I$. Minimal prime ideals are $d$-ideals.

**Proposition 3.2.** $\tau_L$ is an isomorphism iff $L$ is basically disconnected.

**Proof.** ($\Rightarrow$) Assume $\tau_L$ is an isomorphism. By [7, Proposition 3.3], it suffices to show that the annihilator of every element of $RL$ is a principal ideal generated by an idempotent. By the current hypothesis, the
composite

\[
\mathcal{O}(\text{Max}(\mathcal{R}L)) \xrightarrow{\varrho_L} \beta L \xrightarrow{\tau_L} \mathcal{O}(\text{Min}(\mathcal{R}L))
\]

is an isomorphism. Let \( \alpha \in \mathcal{R}L \). Since \( \mathcal{U}(\alpha) \) is complemented, the element of \( \mathcal{O}(\text{Max}(\mathcal{R}L)) \) mapped to it by the isomorphism \( \tau_L \cdot \varrho_L \) is complemented, and so, by the result of Woodward cited above, there is an idempotent \( \eta \in \mathcal{R}L \) such that \( \tau_L \varrho_L (\mathcal{M}(\eta)) = \mathcal{U}(\alpha) \). It is clear that \( I_{(\eta)} = r_L(\text{coz } \eta) \), so that \( \mathcal{U}(\alpha) = \mathcal{U}(O^r_L(\text{coz } \eta)) \). Since \( \text{coz } \eta \) is complemented (as \( \eta \) is an idempotent), \( \eta \in r_L(\text{coz } \eta) \), and hence, for any \( P \in \text{Min}(\mathcal{R}L) \),

\[
\eta \notin P \iff O^r_L(\text{coz } \eta) \notin P.
\]

Thus, \( \mathcal{U}(\alpha) = \mathcal{U}(\eta) \), and hence, by [13, Theorem 2.7],

\[
\text{Ann}(\alpha) = \bigcap \mathcal{U}(\alpha) = \bigcap \mathcal{U}(\eta) = \text{Ann}(\eta) = \langle 1 - \eta \rangle.
\]

Since \( 1 - \eta \) is an idempotent, we are done.

\((\Leftarrow)\) We need only show that \( \tau_L \) is surjective. Because the elements \( \mathcal{U}(\alpha) \) form a base for \( \mathcal{O}(\text{Min}(\mathcal{R}L)) \), we shall be done if we show that each such element has something mapped to it. Now, since minimal prime ideals are \( d \)-ideals, for any \( \alpha \in \mathcal{R}L \) and minimal prime ideal \( P \) of \( \mathcal{R}L \), we have

\[
\alpha \notin P \iff \text{Ann}^2(\alpha) \notin P,
\]

so that, in light of \( \text{Ann}^2(\alpha) = M_{(\text{coz } \alpha)^{**}} = O^r_L((\text{coz } \alpha)^{**}) \), as \( (\text{coz } \alpha)^{**} \) is
complemented since $L$ is basically connected,

$$\alpha \notin P \iff O_{RL}(\text{coz } \alpha)^{**} \notin P.$$  

This implies $\tau_L(O_{RL}(\text{coz } \alpha)^{**}) = \mathcal{U}(\alpha)$, which shows that $\tau_L$ is onto, and is therefore an isomorphism.

Recall that a homomorphism $h: L \to M$ is said to be $*$-dense [12] if, for any $b \in M$, $h_*(b) = 0$ implies $b = 0$. This captures, in a slightly more general form, the notion of a surjective continuous function $f: X \to Y$ being irreducible, in the sense that $f[K] = Y$ for any closed $K \subseteq X$ implies $K = X$.

**Proposition 3.3.** $\tau_L$ is $*$-dense.

**Proof.** We first calculate the right adjoint of $\tau_L$. With the notation as above, note that, for any ideal $P$ of $RL$,

$$I_P = \bigcup \{ r_L(\text{coz } \alpha) \mid \alpha \in P \}$$

because the join defining $I_P$ is directed. We show that $\tau_L(I_P) \subseteq \mathcal{U}(P)$. To start, observe that $O^{I_P} \subseteq P$. Indeed, let $\alpha \in O^{I_P}$. Then $\text{coz } \alpha \in I_P$, implying $\text{coz } \alpha \prec\prec \text{coz } \beta$ for some $\beta \in P$. By [5, Lemma 4.4], this implies $\alpha$ is a multiple of $\beta$, whence $\alpha \in P$. Therefore

$$\tau_L(I_P) = \mathcal{U}(O^{I_P}) \subseteq \mathcal{U}(P).$$

Now, given any ideal $Q$ of $RL$, let $\bar{Q}$ be the subset of $RL$ defined by

$$\bar{Q} = \bigcup \{ T \mid T \text{ is an ideal of } RL \text{ with } \mathcal{U}(T) = \mathcal{U}(Q) \}.$$
The collection whose union is computed is directed because \( U(T_1) = U(T_2) = U(Q) \) implies \( U(T_1 + T_2) = U(Q) \). Thus, \( \bar{Q} \) is an ideal, and, in fact, the largest ideal of \( \mathcal{RL} \) with \( U(\bar{Q}) = U(Q) \). We claim that

\[(\tau_L)_*(U(Q)) = I_{\bar{Q}}.\]

As observed above, \( \tau_L(I_{\bar{Q}}) \subseteq U(\bar{Q}) = U(Q) \). Consider any \( J \in \beta L \) with \( \tau_L(J) \subseteq U(Q) \). Then \( U(O^J) \subseteq U(Q) \), which implies

\[O^J \subseteq O^J + Q \subseteq \bar{Q}.\]

Now let \( a \in J \) and take a \( \gamma \in \mathcal{RL} \) such that \( a \lll \text{coz} \gamma \in J \). Then \( \gamma \in O^J \subseteq \bar{Q} \), which shows that \( a \in I_{\bar{Q}} \). Therefore \( J \subseteq I_{\bar{Q}} \), and hence \( (\tau_L)_*(U(Q)) = I_{\bar{Q}} \), as claimed.

Suppose now that \( U(Q) \) is such that \( (\tau_L)_*(U(Q)) = 0_{\beta L} \). Then \( I_{\bar{Q}} = 0_{\beta L} \), which, by complete regularity, implies \( \bar{Q} = \{0\} \), and hence \( U(Q) = U(\bar{Q}) = \perp \). So \( \tau_L \) is \(*\)-dense. \( \square \)

In the introduction we recalled the Stone extension \( h^\beta : \beta L \to \beta M \) of a frame homomorphism \( h : L \to M \). We remind the reader that, because of the way we view \( \beta L \), the map \( h^\beta \) is given by

\[h^\beta(I) = \{c \in \text{Coz } M \mid c \leq h(d) \text{ for some } d \in I\}.

In light of the above, we have the wedge
which we would like to complete into a commutative square by filling in a homomorphism, say $h: \mathcal{O}(\text{Min}(RL)) \to \mathcal{O}(\text{Min}(RM))$, induced by $h$. We shall need to restrict the map $h$ by requiring that the inverse image of any minimal prime ideal of $RM$ under the ring homomorphism $Rh: RL \to RM$ be minimal prime. This might sound too stringent, but observe that any homomorphism out of a $P$-frame has this property because $L$ is a $P$-frame if and only if every prime ideal of $RL$ is minimal prime [5, Proposition 4.9].

Let us introduce the following notation. Given a homomorphism $h: L \to M$ and an ideal $Q$ of $RL$, we set

$$Q_{(h)} = \{\gamma \in RM \mid \text{coz } \gamma \leq h(\text{coz } \alpha) \text{ for some } \alpha \in Q\}.$$
Since \( T \) is a \( z \)-ideal and \( \beta \notin T \), we must have \( Rh(\alpha) \notin T \), whence \( \alpha \notin (Rh)^{-1}[T] \). Therefore \( Q \notin (Rh)^{-1}[T] \). Conversely, if \( \gamma \) is in \( Q \) but not in \( (Rh)^{-1}[T] \), then \( Rh(\gamma) \) is in \( Q_{(h)} \) but not in \( T \), showing that \( Q_{(h)} \notin T \).

In what follows we use subscripts on \( \mathcal{U} \) to indicate the frame with reference to which the collection of minimal prime ideals is being contemplated. Let \( h: L \to M \) be a balanced homomorphism. Define

\[
\tilde{h}: \mathcal{D}(\text{Min}(\mathcal{R}L)) \to \mathcal{D}(\text{Min}(\mathcal{R}M)) \quad \text{by} \quad \tilde{h}(\mathcal{U}_L(Q)) = \mathcal{U}_M(Q_{(h)}).
\]

Since \( \mathcal{U}_L(Q) \) is not uniquely determined by \( Q \), we must check that \( \tilde{h} \) is a well-defined function. Suppose \( \mathcal{U}_L(Q) = \mathcal{U}_L(R) \) for some ideals \( Q \) and \( R \) in \( \mathcal{R}L \). Let \( T \in \mathcal{U}_M(Q_{(h)}) \). Then \( Q_{(h)} \notin T \), so that, by the lemma above, \( Q \notin (Rh)^{-1}[T] \), whence \( R \notin (Rh)^{-1}[T] \), thence \( R_{(h)} \notin T \). Therefore \( \mathcal{U}_M(Q_{(h)}) \subseteq \mathcal{U}_M(R_{(h)}) \), and hence equality by symmetry.

**Proposition 3.5.** Let \( h: L \to M \) be a balanced homomorphism. The map \( \tilde{h} \) is a frame homomorphism making the square

\[
\begin{array}{ccc}
\beta L & \overset{h^\beta}{\longrightarrow} & \beta M \\
\tau_L \downarrow & & \tau_M \downarrow \\
\mathcal{D}(\text{Min}(\mathcal{R}L)) & \overset{\tilde{h}}{\longrightarrow} & \mathcal{D}(\text{Min}(\mathcal{R}M))
\end{array}
\]

commute. If \( L \) is basically disconnected, then \( \tilde{h} \) is unique with this property.

**Proof.** It is immediate that \( \tilde{h} \) preserves the bottom and the top. An easy application of Lemma 3.4 shows that \( \tilde{h} \) preserves order. We show that \( \tilde{h} \) preserves binary meets. Consider any two ideals \( P \) and \( Q \) in \( \mathcal{R}L \). It
suffices to show that

\[ \overline{h}( \mathcal{U}_L(P) ) \cap \overline{h}( \mathcal{U}_L(Q) ) \subseteq \overline{h}( \mathcal{U}_L(P) \cap \mathcal{U}_L(Q) ) = \overline{h}( \mathcal{U}_L(PQ) ) . \]

Let \( T \in \overline{h}( \mathcal{U}_L(P) ) \cap \overline{h}( \mathcal{U}_L(Q) ) \). Then \( P_{(h)} \not\in T \) and \( Q_{(h)} \not\in T \), which, by Lemma 3.4, implies \( P \not\in (R h)^{-1}[T] \) and \( Q \not\in (R h)^{-1}[T] \), so that \( PQ \not\in (R h)^{-1}[T] \), since \((R h)^{-1}[T]\) is a prime ideal. Consequently, \((PQ)_{(h)} \not\in T\), and hence \( T \in \overline{h}( \mathcal{U}_L(PQ) ) \). Therefore \( \overline{h} \) preserves binary meets.

Next, let \( \{ \mathcal{U}_L(Q_i) \mid i \in I \} \) be a collection of elements of \( \mathcal{O}(\Min(\mathcal{R}L)) \).

We aim to show that \( \overline{h}( \bigvee_i \mathcal{U}_L(Q_i) ) \leq \bigvee_i \overline{h}( \mathcal{U}_L(Q_i) ) \). Put \( P = \sum Q_i \). Then

\[ \overline{h}( \bigvee_i \mathcal{U}_L(Q_i) ) = \overline{h}( \bigcup_i \mathcal{U}_L(Q_i) ) = \overline{h}( \mathcal{U}_L(P) ) = \mathcal{U}_M(P_{(h)}) . \]

Let \( T \in \mathcal{U}_M(P_{(h)}) \). Then, by Lemma 3.4, \( \sum Q_i \not\in (R h)^{-1}[T] \), which implies that there is an index \( i_0 \in I \) for which \( Q_{i_0} \not\in (R h)^{-1}[T] \), so that \((Q_{i_0})_{(h)} \not\in T\). Consequently,

\[ T \in \mathcal{U}_M((Q_{i_0})_{(h)}) \subseteq \bigcup_i \mathcal{U}_M((Q_i)_{(h)}) = \bigvee_i \overline{h}( \mathcal{U}_L(Q_i) ) . \]

Therefore \( \overline{h} \) is a frame homomorphism.

To show that the square commutes, let \( I \in \beta L \). Then

\[ \overline{h}_\mathcal{T}_L(I) = \overline{h}( \mathcal{U}_L(O^I) ) = \mathcal{U}_M(O^I_{(h)}) , \]

and

\[ \mathcal{T}_M \overline{h}^\beta(I) = \mathcal{U}_M(O^{h^\beta(I)}) . \]
We finish the proof by showing that $O_{(h)}^I = O^{h^\beta(I)}$. Let $\gamma \in O_{(h)}^I$. Then $\text{coz}\gamma \leq h(\text{coz}\alpha)$ for some $\alpha \in O^I$. But $\alpha \in O^I$ implies $\text{coz}\alpha \in I$, so that $\text{coz}\gamma \in h^\beta(I)$, whence $\gamma \in O^{h^\beta(I)}$. Therefore $O_{(h)}^I \subseteq O^{h^\beta(I)}$.

On the other hand, let $\sigma \in O^{h^\beta(I)}$. Then $\text{coz}\sigma \in h^\beta(I)$, which implies $\text{coz}\sigma \leq h(\text{coz}\mu)$ for some $\mu$ with $\text{coz}\mu \in I$. Thus $\mu \in O^I$, and therefore $\sigma \in O_{(h)}^I$.

Now suppose $L$ is basically disconnected and that $g: \mathfrak{O}(\text{Min}(\mathcal{R}L)) \to \mathfrak{O}(\text{Min}(\mathcal{R}M))$ satisfies $g \cdot \tau_L = \tau_M \cdot h^\beta$. Then $g \cdot \tau_L = \bar{h} \cdot \tau_L$, and hence $g = \bar{h}$ because $\tau_L$ is an isomorphism by Proposition 3.2. 

\begin{remark}
In [8] it is shown that, for a surjective frame homomorphism $h: L \to M$, the ring homomorphism $\mathcal{R}h: \mathcal{R}L \to \mathcal{R}M$ contracts maximal ideals to maximal ideals if and only if, for every $c \in \text{Coz} L$ and $d \in \text{Coz} M$ with $h(c) \lor d = 1$, there is a $u \in \text{Coz} L$ such that $u \lor c = 1$, and $h(u) \leq d$.

We have not determined if there is such an element-wise characterisation for balanced maps.
\end{remark}

\section{Concluding observations regarding $\text{Min}(\mathcal{R}L)$}

It is shown in [13] that, for any Tychonoff space $X$, $\text{Min}(C(X))$ is countably compact, and it is compact and basically disconnected precisely when every open set is dense in some cozero-set. We conclude by demonstrating that the same results hold for frames.

A ring $A$ is said to satisfy the countable annihilator condition [13], or is called a c.a.c. ring, if for any sequence $(a_n)$ in $A$, there is an $x \in A$ such that $\text{Ann}(x) = \bigcap_{n=1}^{\infty} \text{Ann}(a_n)$. It is observed in [6] that $\mathcal{R}L$ is a c.a.c. ring. Consequently, in view of [13, Theorem 4.9], we have the following result.
**Proposition 4.1.** $\text{Min}(\mathcal{RL})$ is countably compact for any completely regular frame $L$.

Following [10], we say $L$ is **cozero approximated** if, for every $x \in L$, there is an $a \in \text{Coz} L$ such that $a^* = x^*$. In spaces this says for every open set $U \subseteq X$, there is a cozero set $V$ of $X$ such that $U = V$. In [11] a space with this property is called **fraction dense**. Theorem 4.4 of [13] states that if $A$ is an a.c. ring (a weaker form of the c.a.c. property), then $\text{Min}(A)$ is compact and extremally disconnected precisely when for every $B \subseteq A$ there is a $y \in A$ such that $\text{Ann}(B) = \text{Ann}(y)$. Now since annihilator ideals of $\mathcal{RL}$ are precisely the ideals $M_{a^*}$ for $a \in L$, and element-annihilators are exactly the ideals $M_{c^*}$, for $c \in \text{Coz} L$, we have the following.

**Proposition 4.2.** $\text{Min}(\mathcal{RL})$ is compact and basically disconnected iff $L$ is cozero approximated.

The ring $\mathcal{RL}$ is an $f$-ring with **bounded inversion**, which is to say every $\alpha \geq 1$ is invertible. The bounded part of $\mathcal{RL}$ is denoted by $\mathcal{R}^*L$. An easy algebraic calculation shows that $\frac{\alpha}{1+|\alpha|} \in \mathcal{R}^*L$ for any $\alpha \in \mathcal{RL}$. Since $\alpha = \frac{\alpha}{1+|\alpha|} \cdot (1 + |\alpha|)$, and $\text{Ann}(1 + |\alpha|)$ is the zero ideal, it follows from [13, Theorem 5.1] that $\text{Min}(\mathcal{R}(\beta L))$ is homeomorphic to $\text{Min}(\mathcal{RL})$. Consequently, $\beta L$ is cozero approximated iff $L$ is cozero approximated.

**Remark 2.** That $\beta L$ is cozero approximated if and only if $L$ is cozero approximated can also be deduced from these two results: (i) if $h: L \to M$ is dense onto and $L$ is cozero approximated, then so is $M$. This is straightforward. (ii) If $h: L \to M$ is dense coz-onto and $M$ is cozero approximated, then so is $L$. To see this, use the fact that if $g: N \to K$ is a dense frame homomorphism, then $x^* = g_* (x^*)$ for every $x \in N$ (see [9, Lemma 3.1]).
Bibliography


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